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Scaling properties of potential fields due to scaling sources

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Abstract. The theoretical power spectrum of the 3-dimensional potential field caused by an arbitrary 3-dimensional source distribution is derived for gravity and magnetic data. A function with scale-invariant features has a power spectrum, which is proportional to the frequency raised to minus the scaling exponent. For scaling source distributions, the power spectrum of the gravity and magnetic field is anisotropic and a specific scaling exponent exists only for lower-dimensional cross sections of the field. We suggest an approach which allows, under certain conditions, to derive the power spectrum of a lower-dimensional subset from the power spectrum of a 3-dimensional function. For the special case where the 3-dimensional function has an isotropic scaling exponent β^{3D}, we confirm a known property, namely that a (3-k)-dimensional subset of the function has a scaling exponent of approximately k less than β^{3D}. This property is not applicable to the anisotropic 3-dimensional fields, but it can be applied to source distributions with isotropic scaling exponent. Summarizing our results, the scaling exponents of the density distribution and the gravity field are related by

$$\begin{aligned}\beta^{3D}_{dens} &= \beta^{2D}_{dens}+1=\beta^{1D}_{dens}+2\\ &= \beta^{xy}_{field}-1=\beta^{x}_{field}=\beta^{y}_{field}=\beta^{z}_{field}\end{aligned}$$

whereas the relationship between the scaling exponents of the susceptibility distribution and the magnetic field reduced to the pole can be stated as follows:

$$\begin{aligned}\beta^{3D}_{susc} &= \beta^{2D}_{susc}+1=\beta^{1D}_{susc}+2=\beta^{xy}_{field}+1\\ &= \beta^{x}_{field}+2=\beta^{y}_{field}+2=\beta^{z}_{field}+2.\end{aligned}$$

Introduction

The concept of scaling geology has been introduced by *Mandelbrot (1967)*. For several reasons, the scaling exponent β of a 3-dimensional density or susceptibility distribution is an important parameter. First of all, β directly contains information on the local geology. For example, *Pilkington and Todoeschuck (1993)* found scaling exponents of 1.32 to 1.96 for sedimentary and from 2.08 to 2.72 for igneous rocks from susceptibility logs.

Besides the possibility of using β itself to distinguish between different rock types, β also plays an important role in standard least squares inversion. An empirical geophysical dataset can be seen as a convolution of the source distribution with a system function *(Dimri, 1992)*. To recover the source distribution from the measured data by least squares inversion, we have to make *a priori* assumptions on the covariance of the source distribution. The scaling exponent provides exactly this information. That the scaling exponent can be a valuable input for inversion has been demonstrated by *Todoeschuck and Jensen(1989)* for the deconvolution of seismograms, as well as by *Gregotski et al. (1991)*, who proposed a deconvolution operator for fractal susceptibility mapping.

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In this paper we first derive the relationship between the 3-dimensional power spectrum of an arbitrary source distribution and the power spectrum of the respective 3-dimensional field.

The scaling exponents of 3-dimensional density and susceptibility distributions have to be derived from lower dimensional cross sections either of the sources (e.g. density and susceptibility logs), or of their respective fields (e.g. well logs, profiles, maps).

It has already been pointed out, that a scaling exponent that describes scale-invariant features of a n-dimensional function is greater approximately by k than a scaling exponent of any $(n-k)$-dimensional subset of the function *(Falconer, 1990; Turcotte, 1992)*. This property can be directly applied to a 3-dimensional source distribution with isotropic scaling exponent. However, the power spectra of gravity and magnetic fields are anisotropic and do not have a specific scaling exponent. By making an assumption on the phases of the Fourier transform, we derive a general relationship between the power spectrum of a function and the power spectrum of its lower-dimensional subsets. This relationship allows to calculate the scaling exponents for the relevant cross sections of the fields.

Gravity being closely related to magnetic fields, we derive the following properties for both kinds of fields. Since the fields as such are not directly accessible by geophysical measurements, we give attention to the specific nature of the measured data.

Gravity Fields

The anomaly potential $T(r)$ of the gravity field caused by a density distribution $\rho(r)$ satisfies Poisson's equation

$$\Delta T(r) = const\ \rho(r) \tag{1}$$

where *const* refers to all terms which are independent of the location $\vec{r}=(x,y,z)$ and the wave vector $\vec{k}=(k_x,k_y,k_z)=(u,v,w)$.

A differentiation in space domain becomes a multiplication with the respective component of the wave vector in frequency domain, thus leading to

$$k^2\tilde{T}(k) = const\ \tilde{\rho}(k) \tag{2}$$

In gravity surveys usually the vertical gradient $g(r)$ of the gravity potential is measured. In frequency domain we can therefore write

$$|\tilde{g}(k)|^2 = k_z^2 |\tilde{T}(k)|^2 = const\ \frac{k_z^2}{k^4}\ |\tilde{\rho}(k)|^2 \qquad (3)$$

This is the relationship between the power spectrum of the vertical gradient of the anomaly potential and the power spectrum of the density distribution.

Magnetic Fields

We will now derive a relationship between the spectrum of the susceptibility distribution and the magnetic field.

Let us first clarify what we are actually measuring with a total intensity magnetometer.

The magnetic field is the *vector* sum of the normal field and the anomaly field caused by the susceptibility distribution. When we subtract the intensity of the normal field from the measured intensity, we only get the component of the anomaly field *parallel* to the normal field, which is equivalent to the derivation of the potential $U_{anomaly}$ of the anomaly field in direction of the normal field. The observed anomaly of intensity of the magnetic field T_a can thus be expressed as

$$T_a(r) = \vec{\nu} \cdot \vec{\nabla} U_{anomaly}(r) \qquad (4)$$

where $\vec{\nu}$ is a unit vector in direction of the normal field.

For the following derivation we will make the assumptions of isotropic susceptibility $\chi(r)$, homogeneous normal field $\vec{N}_0$ and negligible remanent magnetization. Under these simplifying assumptions the magnetization in a specific location can be directly obtained from the susceptibility in this point. Furthermore, the potential $U_{anomaly}$ can then be expressed as the potential of a field of magnetic dipoles $\chi(r')\vec{N}_0$:

$$\begin{aligned} U_{anomaly}(r) &= const \int \chi(r')\vec{N}_0 \cdot \vec{\nabla}'\,\frac{1}{|\vec{r}-\vec{r}'|}\,d^3r' \\ &= -const \int \chi(r')\vec{N}_0 \cdot \vec{\nabla}\,\frac{1}{|\vec{r}-\vec{r}'|}\,d^3r' \\ &= -const\ \vec{N}_0 \cdot \vec{\nabla} \int \chi(r')\,\frac{1}{|\vec{r}-\vec{r}'|}\,d^3r' \\ &= -const\ \vec{N}_0 \cdot \vec{\nabla} V(r) \qquad (5) \end{aligned}$$

where V(r) has the properties of a virtual gravity potential, the susceptibility taking on the role of a virtual density. The gradient of the anomaly potential is thus

$$\begin{aligned} \nabla U_{anomaly}(r) &= const\ (\vec{N}_0 \cdot \vec{\nabla})\,\vec{\nabla} V(r) \\ &\quad + const\ \vec{N}_0 \times \vec{\nabla} \times \vec{\nabla} V(r) \\ &= const\ (\vec{N}_0 \cdot \vec{\nabla})\,\vec{\nabla} V(r) \qquad (6) \end{aligned}$$

For the anomaly of the intensity of the magnetic field we then find

$$\begin{aligned} T_a(r) &= \vec{\nu} \cdot \vec{\nabla} U_{anomaly}(r) \\ &= const\ (\vec{N}_0 \cdot \vec{\nabla})(\vec{N}_0 \cdot \vec{\nabla} V(r)) \qquad (7) \end{aligned}$$

Transformation to frequency domain yields

$$\begin{aligned} \tilde{T}_a(k) &= const\ (\vec{N}_0 \cdot \vec{k})(\vec{N}_0 \cdot \vec{k})\ \tilde{V}(k) \\ &= const\ \frac{(\vec{N}_0 \cdot \vec{k})^2}{k^2}\ \tilde{\chi}(k) \\ |\tilde{T}_a(k)|^2 &= const\ \frac{(\vec{N}_0 \cdot \vec{k})^4}{k^4}\ |\tilde{\chi}(k)|^2 \qquad (8) \end{aligned}$$

Equation (8) shows that the power spectrum of T_a is anisotropic and depends on the direction of the normal field $\vec{N}_0$.

Let us assume now, that the field has been reduced to the pole, which is possible with the above assumptions. Then $\vec{N}_0 = (0, 0, 1)$ and we get

$$|\tilde{T}_a(k)|^2 = const\ \frac{k_z^4}{k^4}\ |\tilde{\chi}(k)|^2 \qquad (9)$$

This is the relationship between the power spectrum of the anomaly of the magnetic field intensity and the power spectrum of the susceptibility distribution.

The Scaling Exponent

Recently, *Pilkington and Todoeschuck (1993)* have shown that the $\beta \approx 3$ power spectra of aeromagnetic data observed by *Gregotski et al. (1991)* imply a 3-dimensional susceptibility distribution with $\beta \approx 4$. This seemed to be inconsistent with the observed $\beta \approx 2$ for susceptibility logs observed by *Pilkington and Todoeschuck (1993)*. To account for this inconsistency *Pilkington and Todoeschuck* postulated anisotropic scaling with $\beta \approx 4$ for the horizontal susceptibility distribution and $\beta \approx 2$ for the vertical distribution.

As we will see, it is not required to postulate anisotropic scaling behaviour to explain the observed scaling exponents. On the contrary, an isotropic 3-dimensional susceptibility distribution with $\beta = 4$ will necessarily lead to the observed $\beta = 2$ for a well log.

This is a special case of the more general question, how the power spectrum of a n-dimensional function is related to the power spectra of lower-dimensional subsets of this function. We will begin with the 2-dimensional case and consider a function $F(x, y) : R^2 \longrightarrow C$ with its Fourier transform $\tilde{F}(u, v) : R^2 \longrightarrow C$.

We want to know the Fourier transform $\tilde{f}(u)$ of a 1-dimensional subset $f(\xi) := F(\xi\cos(\varphi), \xi\sin(\varphi))$ of $F(x, y)$ in an arbitrary direction φ.

To facilitate the following derivations, we will assume $\varphi = 0$, taking $f(x)$ as the values of F along the x-axis ($f(x) = F(x, 0)$).

$|\tilde{F}(u, v)|$ can be seen as the amplitude of a planar wave travelling in (u, v)-direction. If this direction is not exactly perpendicular to the x-axis, then this wave will contribute with its full intensity to $\tilde{f}(u)$.

Let us consider a fixed frequency u_0. Then we have to add all 2D-waves with direction (u_0, v) and amplitude and phase $\tilde{F}(u_0, v)$ to get $\tilde{f}(u_0)$. To add two waves, we have to know their phase shift. If it is zero, the amplitudes *add*, whereas for 180° phase shift we have to *subtract* the amplitudes. This property is expressed in the following inequality:

$$0 \le |\tilde{f}(u)| \le \int_{-\infty}^{\infty} |\tilde{F}(u,v)| dv \quad (10)$$

Let us assume that there is no relationship between the phases, i.e. the waves are incoherent. Then we have to add the intensities instead of the amplitudes. In this case we get

$$|\tilde{f}(u)|^2 = \int_{-\infty}^{\infty} |\tilde{F}(u,v)|^2 dv \quad (11)$$

which relates the 1D-power spectrum of the subset to the 2D-power spectrum of the superset.

In the special case where $F(x, y)$ has an isotropic scaling exponent we can write

$$|\tilde{F}(u,v)|^2 \propto \sqrt{u^2+v^2}^{\,-\beta} \quad (12)$$

To calculate the 1D-power spectrum of the subset $f(x)$ we have to solve the integral

$$\begin{aligned} |\tilde{f}(u)|^2 &= \int_{-\infty}^{\infty} \sqrt{u^2+v^2}^{\,-\beta} dv \\ &= \int_{-\infty}^{\infty} u^{-\beta} \sqrt{1+\frac{v^2}{u^2}}^{\,-\beta} dv \end{aligned} \quad (13)$$

Substituting $a = \frac{v}{u}$, $(u \neq 0)$ gives $dv = u\, da$ and

$$\begin{aligned} |\tilde{f}(u)|^2 &= \int_{-\infty}^{\infty} u^{-\beta}\, u \sqrt{1+a^2}^{\,-\beta} da \\ &= u^{-\beta+1} \int_{-\infty}^{\infty} \sqrt{1+a^2}^{\,-\beta} da \end{aligned} \quad (14)$$

For $\beta \ge 2$ the integral which is now independent of u converges and

$$|\tilde{f}(u)|^2 = const\; u^{-\beta+1} \quad (15)$$

Therefore, the same function $F(x, y)$ having a β power spectrum in two dimensions will have a $\beta - 1$ power spectrum for any 1-dimensional subset.

These derivations can be easily generalized to higher dimensions on substituting u^2 by $u^2 + w^2$, yielding

$$\beta^{1D} = \beta^{2D} - 1 = \beta^{3D} - 2 \quad (16)$$

This basic property of β, which is not restricted to potential fields, has already been pointed out for instance by Turcotte (1992). He, however, has found (16) to be an approximation rather than an equation. The reason why relationship (16) does not have to be exact is, that we have assumed incoherency of the planar waves that we added to obtain (11). Coming back to inequality (10) and solving the integral for a scaling function yields

$$0 \le |\tilde{f}(u)|^2 \le const\; u^{-\beta+2} \quad (17)$$

Therefore, the difference between the scaling exponents β^{1D} and β^{2D} can be more than one, if the waves added tend to be in phase. Turcotte (1992, Table 7.2) found differences of 1.02 to 1.09 between the 1-dimensional and 2-dimensional scaling exponents for topographic data. As far as we are aware, a geological explanation of these deviations has not been attempted. To see whether such deviations can also be observed for potential fields, we analyzed a high quality aeromagnetic survey. In our case the scaling exponent of the 2-dimensional field was exactly one (± 0.05) more than the scaling exponent of the average power spectrum along the flight lines.

In any case, equation (16) seems to be at least a good approximation, which allows for example, to derive the isotropic scaling exponent β^{3D}_{source} from the observable β^{1D}_{source} of a density or susceptibility well log.

Scaling Exponent of the Gravity Field

Using equation (3) we can now calculate the observable scaling exponents of lower dimensional cross sections of the gravity field. We will start by deriving the scaling exponent β_{xy} observed in the horizontal observation plane:

$$|\tilde{g}_{xy}(k_x, k_y)|^2 = c \int_{-\infty}^{\infty} k_z^2 \sqrt{k_x^2+k_y^2+k_z^2}^{\,-\beta^{3D}_{dens}-4} dk_z$$

where *const* has been abbreviated by c. Replacing $k_{xy} = \sqrt{k_x^2+k_y^2}$ we get

$$|\tilde{g}_{xy}|^2 = c\, k_{xy}^{-\beta^{3D}_{dens}-2} \int_{-\infty}^{\infty} \frac{k_z^2}{k_{xy}^2} \sqrt{1+\frac{k_z^2}{k_{xy}^2}}^{\,-\beta^{3D}_{dens}-4} dk_z$$

Substituting $a = \frac{k_z}{k_{xy}}$ gives $dk_z = k_{xy}\, da$ and

$$|\tilde{g}_{xy}|^2 = c\, k_{xy}^{-\beta^{3D}_{dens}-2}\, k_{xy} \int_{-\infty}^{\infty} a^2 \sqrt{1+a^2}^{\,-\beta^{3D}_{dens}-4} da$$

For $\beta^{3D}_{dens} \ge 0$ the integral converges and

$$|\tilde{g}_{xy}|^2 = c\, k_{xy}^{-\beta^{3D}_{dens}-1} = c \sqrt{k_x^2+k_y^2}^{\,-(\beta^{3D}_{dens}+1)} \quad (18)$$

In the horizontal observation plane a gravity survey will therefore yield $\beta^{xy}_{field} = \beta^{3D}_{dens} + 1$.

Since β^{xy}_{field} is isotropic, we immediately get the scaling exponents for profiles:

$$\beta^{x}_{field} = \beta^{y}_{field} = \beta^{xy}_{field} - 1 = \beta^{3D}_{dens} \quad (19)$$

For β^{z}_{field}, a similar calculation yields $\beta^{z}_{field} = \beta^{3D}_{dens}$.

Scaling Exponent of the Magnetic Field

Using equation (9) the following relationship for a field which has been reduced to the pole is obtained in a similar way:

$$\begin{aligned} \beta^{3D}_{susc} &= \beta^{xy}_{field} + 1 = \beta^{x}_{field} + 2 \\ &= \beta^{y}_{field} + 2 = \beta^{z}_{field} + 2 \end{aligned} \quad (20)$$

This result is consistent with the work of *Pilkington and Todoeschuck (1993)* who found the same relationship between β^{3D}_{susc} and β^{xy}_{field} using a formula of *Naidu (1968)*.

Conclusion

We have shown, in which way the scaling exponents of gravity and magnetic fields are related to the scaling exponents of their source distributions. Further, we have shown how β changes with dimension. The results are summarized in table 1. It is interesting to note that for any 1-dimensional cross section of a particular field the scaling exponents are equal, despite the fact that the 3-dimensional power spectrum of the field is anisotropic.

Table 1. Relationships between the scaling exponents.

	Source			**Field**	
	3D	2D	1D	XY	1D
Gravity	β	$\beta-1$	$\beta-2$	$\beta+1$	β
Magnetic	β	$\beta-1$	$\beta-2$	$\beta-1$	$\beta-2$

In the section on the scaling exponent we have pointed out that in some cases these relationships may be only approximately correct. But so far we have not found a significant deviation from the predicted values, neither for synthetic nor for real data. Nevertheless, geological situations may exist, where these relationships are not exact. This could be an interesting topic for further research.

In this paper all derivations were made in cartesian coordinates. Interesting new results may be obtained by considering the spherical case. This may be particularly important due to the available global data sets of satellite altimetry and magsat anomalies. The gravity field in spherical coordinates has been studied for instance by *Khan (1977)* in a different context.

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